

# THE NAIVE APPROACH FOR CONSTRUCTING THE DERIVED CATEGORY OF A $d$ -ABELIAN CATEGORY FAILS

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**ABSTRACT.** Let  $k$  be a field. In this short note we give an example of a 2-abelian  $k$ -category, realized as a 2-cluster-tilting subcategory of the category  $\mathbf{mod} A$  of finite dimensional (right)  $A$ -modules over a finite dimensional  $k$ -algebra  $A$ , for which the naive idea for constructing its “bounded derived category” as 2-cluster-tilting subcategory of the bounded derived category of  $\mathbf{mod} A$  cannot work.

## INTRODUCTION

Let  $d$  be a positive integer. Motivated by Iyama’s higher Auslander–Reiten theory [Iya07], the class of  $d$ -abelian categories was introduced in [Jas16] as an abstract framework for investigating the intrinsic properties of  $d$ -cluster-tilting subcategories of abelian categories. Note that  $d$ -cluster-tilting subcategories of abelian categories are  $d$ -abelian categories, see [Jas16, Thm. 3.16]. If for a  $d$ -abelian category the projective and injective objects coincide, that is it is a *Frobenius  $d$ -abelian category*, then its stable category is a  $(d + 2)$ -angulated category in the sense of [GKO13]. It is then a natural problem to construct a “derived category” of a given  $d$ -abelian category as well, which is expected to be a  $(d + 2)$ -angulated category.

There is the following evidence in support of this expectation. Given a  $d$ -representation-finite algebra  $A$  in the sense of [IO11], it is known that there is a unique  $d$ -cluster-tilting subcategory  $\mathcal{M}(A)$  of  $\mathbf{mod} A$ , the category of finite dimensional (right)  $A$ -modules, see [Iya11, Prop. 1.3]. We remind the reader that  $d$ -representation-finite algebras are to be thought of as analogues of hereditary algebras of finite representation type from the viewpoint of higher Auslander–Reiten theory. Moreover, the bounded derived category of  $\mathbf{mod} A$ , denoted by  $D^b(\mathbf{mod} A)$ , has a  $d$ -cluster-tilting subcategory

$$\mathcal{U}(A) := \text{add} \{ M[di] \in D^b(\mathbf{mod} A) \mid M \in \mathcal{M} \text{ and } i \in \mathbb{Z} \}$$

which is closed under the  $d$ -th power of the shift functor of  $D^b(\mathbf{mod} A)$ , see [Iya11, Thm. 1.21]. Hence, it follows from [GKO13, Thm. 1] that  $\mathcal{U}(A)$  is a  $(d + 2)$ -angulated category. The category  $\mathcal{U}(A)$  can be thought of as a “bounded derived category” of  $d$ -abelian category  $\mathcal{M}(A)$ . Note that this case is particular in that every complex in  $\mathcal{U}(A)$  is isomorphic to a finite direct sum of stalk complexes, reflecting the “hereditary nature” of  $\mathcal{M}(A)$ .

In this short note, we give an example of a finite dimensional algebra  $A$  of global dimension 4 such that  $\mathbf{mod} A$  has a unique 2-cluster-tilting subcategory  $\mathcal{M}(A)$ . However, the only rigid subcategory of  $D^b(\mathbf{mod} A)$  containing  $\mathcal{M}(A)$  and which is closed under the second power of the shift functor of  $D^b(\mathbf{mod} A)$  is precisely  $\mathcal{U}(A)$  as defined above. In this case, it turns out that  $\mathcal{U}(A)$  is neither a 2-cluster-tilting subcategory of  $D^b(\mathbf{mod} A)$  nor is it a 4-angulated category with suspension induced by the second power of the shift of  $D^b(\mathbf{mod} A)$ . This shows that if there is a

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*Date:* April 13, 2016.

*2010 Mathematics Subject Classification.* Primary: 16G70. Secondary: 16G20.

“bounded derived category” of the 2-abelian  $k$ -category  $\mathcal{M}(A)$ , then it cannot be realized as a 2-cluster-tilting subcategory of  $D^b(\text{mod } A)$ .

#### THE EXAMPLE

Let  $Q$  be the quiver  $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$  and  $A$  be the quotient of the path algebra  $kQ$  by the ideal generated by all paths of length 2 in  $Q$ . It is well known that the algebras  $kQ$  and  $A$  are derived equivalent. The Auslander–Reiten quiver of  $\text{mod } A$  is the following:

$$\begin{array}{ccccccc} & 4 & & 3 & & 2 & & 1 \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow \\ 5 & \cdots & 4 & \cdots & 3 & \cdots & 2 & \cdots & 1 \end{array}$$

It is straightforward to verify that

$$\mathcal{M}(A) := \text{add} \left( 5 \oplus \frac{4}{5} \oplus \frac{3}{4} \oplus 3 \oplus \frac{2}{3} \oplus \frac{1}{2} \oplus 1 \right)$$

is the unique 2-cluster-tilting subcategory of  $\text{mod } A$ , see e.g. [Jas16, Prop. 6.2]. It is also clear that  $\Omega^2(\mathcal{M}) \subset \mathcal{M}$ , where  $\Omega$  is Heller’s syzygy functor.

**Proposition.** *Let  $\mathcal{U}$  be a rigid subcategory of  $D^b(\text{mod } A)$  containing  $\mathcal{M}(A)$  and satisfying  $\mathcal{U}[2] = \mathcal{U}$ . Then,*

$$\mathcal{U} = \mathcal{U}(A) := \text{add} \{ M[2i] \in D^b(\text{mod } A) \mid M \in \mathcal{M}(A) \text{ and } i \in \mathbb{Z} \}.$$

Moreover,  $\mathcal{U}(A)$  is neither a 2-cluster-tilting subcategory of  $D^b(\text{mod } A)$  nor it is a 4-angulated category with suspension induced by the second power of the shift of  $D^b(\text{mod } A)$ .

*Proof.* The first claim is a straightforward verification. We provide the Auslander–Reiten quiver of  $D^b(\text{mod } A)$  in Figure 1 for the convenience of the reader.

Suppose that  $\mathcal{U}(A)$  is a 4-angulated category with respect to the second power of the shift functor of  $D^b(\text{mod } A)$  and consider the morphism  $f: \frac{3}{4} \rightarrow \frac{2}{3}$ . By assumption, there exist a 4-angle

$$\frac{3}{4} \xrightarrow{f} \frac{2}{3} \xrightarrow{g} X \xrightarrow{h} Y \xrightarrow{i} \frac{3}{4}[2].$$

Since  $f$  is not a retraction, [GKO13, Prop. 2.5 (a)] implies that the morphism  $g$  is non-zero. Taking a minimal version of  $g$ , we may assume that  $X = \frac{1}{2}$ . Repeating the same argument for  $g$ , we deduce that  $h$  is non-zero and  $Y = 1$ . But there are no non-zero homomorphisms  $1 \rightarrow \frac{2}{3}[2]$  since  $\frac{2}{3}$  is an injective  $A$ -module. Hence  $i = 0$ , but this implies that  $f$  is a section, a contradiction. This shows that  $\mathcal{U}(A)$  cannot be endowed with the structure of a 4-angulated category for which the suspension is given by the second power of the shift functor of  $D^b(\text{mod } A)$ . In particular, it cannot be a 2-cluster-tilting subcategory of  $D^b(\text{mod } A)$  as this would contradict [GKO13, Thm. 1].  $\square$

*Remark.* Note that one can also see that  $\mathcal{U}(A)$  is not a 2-cluster-tilting subcategory of  $D^b(\text{mod } A)$  by observing that

$$\text{Ext}_A^1 \left( \frac{3}{4} \rightarrow \frac{2}{3} \rightarrow \frac{1}{2}, 3 \right) \neq 0$$

while

$$\text{Ext}_A^1 \left( \mathcal{U}(A), \frac{3}{4} \rightarrow \frac{2}{3} \rightarrow \frac{1}{2} \right) = 0.$$

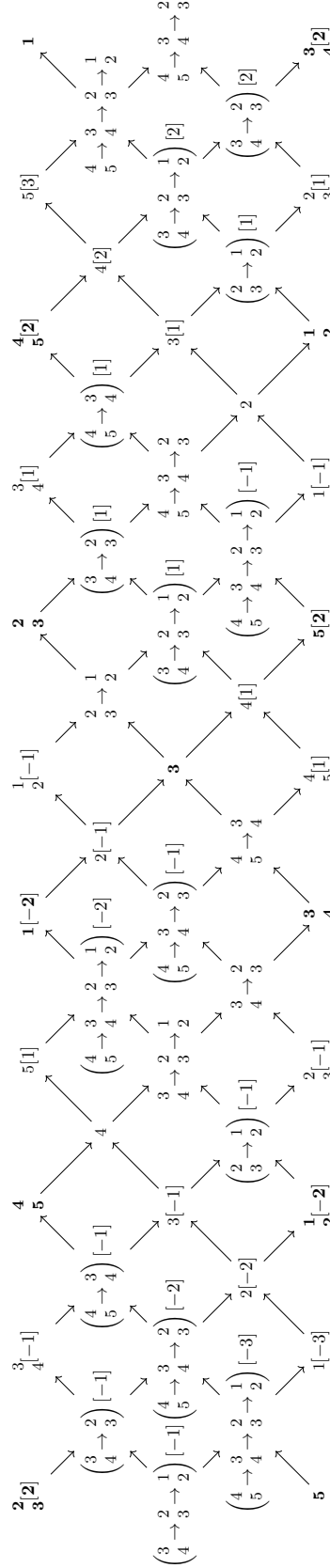


FIGURE 1. Auslander-Reiten quiver of  $D^b(\text{mod } A)$ . The subcategory  $\mathcal{U}(A)$  is indicated with bold face.

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